APPROXIMATE SOLUTION OF SOME NONLINEAR PROBLEMS OF HEAT CONDUCTION THEORY

A. N. Luppov and B. G. Ogloblin

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In this paper we obtain approximate solutions for the problem of unsteady heat conduction in a heat-radiating plate, a solid cylinder, a hollow cylinder, and a sphere, in which internal heat sources are present. The solution involves the assumption of a parabolic temperature profile.

The investigation of heat transfer processes in a solid with radiant heat exchange at its surface leads (see [1]) to the heat-conduction equation with the nonlinear boundary condition

$$-\lambda \frac{\partial T}{\partial n}\bigg|_s = \sigma\varepsilon\left(T|_s^4 - \vartheta^4\right).$$

Existence and uniqueness theorems have been proved (see [2]) for boundary problems of this type; however, an exact analytic solution, even for the simplest regions, is as yet unknown. There is, in this regard, considerable interest in formulating an approximate solution of individual problems of the type mentioned. For bodies without internal heat sources, such solutions have been obtained by diverse methods; see, for example, [3-5].

In this paper we obtain solutions for bodies of simplest geometric form with internal heat generation and with radiant heat exchange at the boundary in which we assume a parabolic temperature profile.

Heating of a Plate, Cylinder, and Sphere. Consider the equation

$$c(T)\gamma(T)\frac{\partial T}{\partial t} = \frac{1}{r^\alpha}\frac{\partial}{\partial r}\left[r^\alpha \lambda(T)\frac{\partial T}{\partial r}\right] + S(t) \quad (1)$$

for the region $0 < r \leq R$; $0 \leq t < \infty$. For $\alpha = 0$, 1, and 2 it describes a symmetric temperature field in a plate, an unbounded cylinder, and a sphere, respectively. For the initial condition we have $T(r, 0) = T_0$. The boundary conditions have the form

$$\frac{\partial T(0,t)}{\partial r} = 0; \quad -\lambda(T(R,t))\frac{\partial T(R,t)}{\partial r} = \sigma\varepsilon\left[T^4(R,t) - \vartheta^4(t)\right]. \quad (2)$$

We shall seek a solution in the form

$$T(r,t) = a_0(t) + a_1(t)r + a_2(t)r^2. \quad (3)$$

We introduce an unknown function p(t), expressing the temperature at the boundary r = R, and using Eqs. (2), (3), we express T(r, t) in terms of p(t). From the first boundary condition it follows that $a_1(t) \equiv 0$, while the second yields

$$2a_2R = -\frac{\sigma\varepsilon}{\lambda(p)}\left(p^4 - \vartheta^4\right).$$

In addition, we have

$$a_0 + a_2R^2 = p,$$

whence

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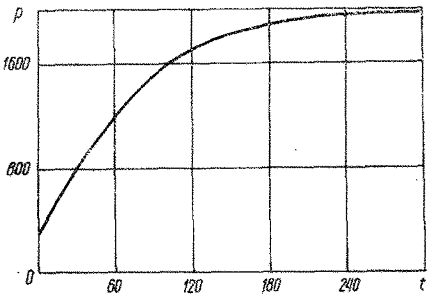

Fig. 1

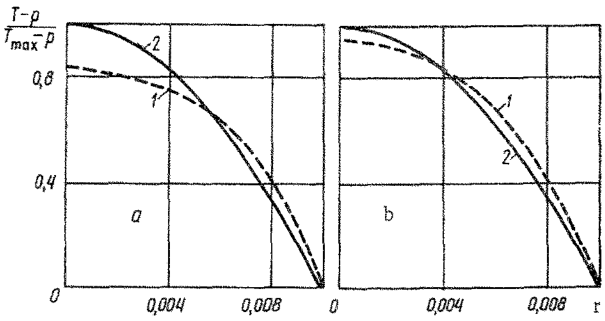

Fig. 2

Fig. 1. Variation of the temperature of the cylinder surface (t in sec, p in °K).

Fig. 2. Temperature distribution along a cylinder radius at (a) t = 17 sec; (b) t = 62 sec. (radius r in meters) Curve 1: solution obtained by numerically integrating Eq. (1); curve 2: solution based on formulas (7), (4).

$$a_2 = -\frac{\sigma\varepsilon(p^4-\vartheta^4)}{2R\lambda(p)};\quad a_0 = p + \frac{\sigma\varepsilon R(p^4-\vartheta^4)}{2\lambda(p)}.$$

Thus the temperature of the body is expressed in terms of the boundary temperature

$$T(r,t) = p + \frac{\sigma\varepsilon R(p^4-\vartheta^4)}{2\lambda(p)}\left(1-\frac{r^2}{R^2}\right). \tag{4}$$

We now multiply both sides of Eq. (1) by r^α and integrate it through with respect to r from 0 to R, making use of Eq. (4). In addition we carry $c(T)\gamma(T)$ across the integral sign as $c(\bar{T})\gamma(\bar{T})$, where $\bar{T}$ is a mean temperature defined in terms of the boundary temperature by the formula

$$\bar{T}(t) = p + \frac{\sigma\varepsilon R(p^4-\vartheta^4)}{(\alpha+3)\lambda(p)}. \tag{5}$$

Upon carrying through the required differentiations with the substitution of T(r, t) from formula (4) into the left member of Eq. (1), the latter having been previously integrated through, we obtain an ordinary differential equation for p(t). Solving this equation for dp/dt, we obtain

$$\frac{dp}{dt} = \frac{S(t) - \frac{(\alpha+1)\sigma\varepsilon}{R}(p^4-\vartheta^4) + \frac{4\sigma\varepsilon R\vartheta^3 c(\bar{T})\gamma(\bar{T})}{(\alpha+3)\lambda(p)}\frac{d\vartheta}{dt}}{\left[1+\frac{4\sigma\varepsilon Rp^3}{(\alpha+3)\lambda(p)} - \frac{\sigma\varepsilon R(p^4-\vartheta^4)}{(\alpha+3)\lambda^2(p)}\frac{d\lambda}{dp}\right]c(\bar{T})\gamma(\bar{T})}. \tag{6}$$

In general Eq. (6) may only be integrated numerically. We note, however, an important special case in which it has an explicit solution.

Assume that the thermophysical parameters c, γ, λ are independent of the temperature and that the specific power S and the temperature of the medium are constant. Equation (6) then becomes

$$\frac{dp}{dt} = \frac{S - \frac{(\alpha+1)\sigma\varepsilon}{R}(p^4-\vartheta^4)}{\left[1+\frac{4\sigma\varepsilon Rp^3}{(\alpha+3)\lambda}\right]c\gamma},$$

which may be integrated in terms of quadratures:

$$\frac{c\gamma R}{4\sigma\varepsilon(\alpha+1)\sqrt[4]{\left[\frac{SR}{(\alpha+1)\sigma\varepsilon}+\vartheta^4\right]^3}}\left\{\ln\left|\frac{p+\sqrt[4]{\frac{SR}{(\alpha+1)\sigma\varepsilon}+\vartheta^4}}{p-\sqrt[4]{\frac{SR}{(\alpha+1)\sigma\varepsilon}+\vartheta^4}}\right| + 2\,\mathrm{arctg}\,\frac{p}{\sqrt[4]{\frac{SR}{(\alpha+1)\sigma\varepsilon}+\vartheta^4}}\right\}$$

$$-\frac{c\gamma R^2}{(\alpha+1)(\alpha+3)\lambda}\ln\left|1-\frac{p^4}{\frac{SR}{(\alpha+1)\sigma\varepsilon}+\vartheta^4}\right| = t + C_1. \tag{7}$$

The constant C_1 may be determined from the condition $p(0) = T_0$. Equation (7), along with Eq. (4), enables us to find the temperature of the body at an arbitrary time instant t. Moreover, depending on α, it gives the temperature in a plate, an unbounded cylinder, or a sphere.

Heating of an Unbounded Hollow Cylinder. Consider Eq. (1) in the region $R_1 \leq r \leq R_2$, $0 \leq t < \infty$ for $\alpha = 1$. In this case it describes the temperature field in an unbounded hollow cylinder. We assume that no heat flow takes place across the boundary $r = R_1$, and that at the boundary $r = R_2$ heat exchange takes place through radiation with a medium at temperature ϑ.

We give only the final results, obtained in a manner similar to the previous results. The formulas in this case correspond to the formulas (4)-(7):

$$T(r, t) = p + \frac{\sigma\varepsilon(p^4 - \vartheta^4) R_1 R_2}{2(R_2 - R_1)\lambda(p)}\left(\frac{R_2}{R_1} - 2 + \frac{2r}{R_2} - \frac{r^2}{R_1 R_2}\right), \tag{4'}$$

$$\overline{T}(t) = p + \frac{\sigma\varepsilon(R_2 - R_1)(5R_1 + 3R_2)(p^4 - \vartheta^4)}{12\lambda(p)(R_1 + R_2)}, \tag{5'}$$

$$\frac{dp}{dt} = \left[S(t) - \frac{2\sigma\varepsilon R_2}{R_2^2 - R_1^2}(p^4 - \vartheta^4) + \frac{c(\overline{T})\gamma(\overline{T})(R_2 - R_1)(5R_1 + 3R_2)\vartheta^3}{3\lambda(p)(R_1 + R_2)}\frac{d\vartheta}{dt}\right]\left\{\left[1 + \frac{(R_2 - R_1)(5R_1 + 3R_2)\sigma\varepsilon p^3}{3\lambda(p)(R_1 + R_2)}\right.\right.$$
$$\left.\left. - \frac{(R_2 - R_1)(5R_1 + 3R_2)\sigma\varepsilon(p^4 - \vartheta^4)}{12\lambda^2(p)(R_1 + R_2)}\frac{d\lambda}{dp}\right]c(\overline{T})\gamma(\overline{T})\right\}^{-1}, \tag{6'}$$

$$\frac{c\gamma(R_2^2 - R_1^2)}{8\sigma\varepsilon R_2\sqrt[4]{\left[\frac{S(R_2^2 - R_1^2)}{2\sigma\varepsilon R_2} + \vartheta^4\right]^3}}\left\{\ln\left|\frac{p + \sqrt[4]{\frac{S(R_2^2 - R_1^2)}{2\sigma\varepsilon R_2} + \vartheta^4}}{p - \sqrt[4]{\frac{S(R_2^2 - R_1^2)}{2\sigma\varepsilon R_2} + \vartheta^4}}\right| + 2\,\mathrm{arctg}\frac{p}{\sqrt[4]{\frac{S(R_2^2 - R_1^2)}{2\sigma\varepsilon R_2} + \vartheta^4}}\right\}$$
$$- \frac{c\gamma(R_2 - R_1)^2(5R_1 + 3R_2)}{4R_2\lambda}\ln\left|1 - \frac{p^4}{\frac{S(R_2^2 - R_1^2)}{2\sigma\varepsilon R_2} + \vartheta^4}\right| = t + C_2. \tag{7'}$$

We note that the constant C_1 (or correspondingly the constant C_2) allows us to satisfy the initial condition at the radiating surface only. However, if $\vartheta(0) = T_0$ the initial condition is satisfied automatically throughout the region. In a more general case, instead of requiring the initial temperature to be constant, it is sufficient to assume that heating occurs from a steady state and that $\vartheta(t)$ has no jump at $t = 0$.

For the hollow cylinder the requirement that the inner surface be insulated is not necessary. With insignificant changes the solution can also be obtained for another boundary condition.

A Numerical Example. As an example we consider heating of an unbounded cylinder of radius $R = 0.01$ m, where the cylinder material has the thermophysical parameters: $c = 250$ J/kg·deg, $\gamma = 10^4$ kg/m^3, $\lambda = 2.9$ W/m·deg. The specific power of the source $S = 4 \cdot 10^7$ W/m^3. Temperature of the surrounding medium $\vartheta = T_0 = 300$°K. Emissivity $\varepsilon = 0.23$. Figure 1 shows the variation of temperature p at the surface of the cylinder. In Fig. 2 temperature distributions along a cylinder radius are shown at two different times, calculations being made using formulas (7), (4) and also by numerically integrating Eq. (1). It is evident that the values of the temperature on the boundary and also the values of the mean temperatures, obtained by these two methods, are practically coincident.

At $r = 0$ the temperatures calculated from formulas (7), (4) are somewhat higher. As the temperature increases the relative value of this error diminishes, and as a stationary thermal regime is reached our solution becomes an exact solution.

Two stages in the heating may be distinguished: a broadening of the zone of influence of the boundary conditions and heating with a parabolic temperature profile throughout the thickness of the body, as was the case, for example, in [4]; in our case this does not give the desired results. In the absence of heat sources formula (7) differs only insignificantly from the formula given in [4] for the second stage of heating. However, in this case a jump in temperature of the surrounding medium takes place at $t = 0$, which makes it necessary to consider two stages in the heating in every case involving large values of $\sigma\varepsilon\vartheta^3 R/\lambda$.

NOTATION

T	is the body temperature;
ϑ	is the temperature of surrounding medium;
t	is the time;
r	is the radial coordinate;
α	is the index determining body geometry;
c	is the specific capacity;
γ	is the density;
λ	is the thermal conductivity;
R	is the thickness (radius) of body;
T_0	is the reference temperature;
σ	is the Stefan-Boltzmann coefficient;
ε	is the emissivity;
p	is the temperature of emitting surface;
S	is the specific power of source;
R_1, R_2	are the internal and external radii of hollow cylinder.

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